q-Analogs of Classical 6-Periodicity: From Euler to Chebyshev

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Abstract

The sequence of period 6 starting with $1, 1, 0, -1, -1, 0$ appears in many different disguises in mathematics. Various q-versions of this sequence are found, and their relations with Euler's pentagonal numbers theorem and Chebyshev polynomials are discussed.

> The motto on Cardinal Newman's tomb ought to be the funeral motto of every Catholic,
> *Ex umbris et imaginibus in veritatem,*
> Out of shadows and appearances into the truth.
>
> Ronald Knox, *The Pastoral Sermons*

1   Introduction

The sequence of period 6, starting with

$$1, 1, 0, -1, -1, 0, \tag{1.1}$$

appears as the coefficients in the expansion

$$\frac{1}{1 - x + x^2} = 1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + x^{12} + x^{13} - \cdots . \tag{1.2}$$

This is easy to see ([15, Ch. V]):

$$\sum_{s=0}^{\infty} a_s x^s = \frac{1}{1 - x + x^2} = \frac{1 + x}{1 + x^3} = \frac{1}{1 + x^3} + x \frac{1}{1 + x^3}$$

$$= \sum_{\ell=0}^{\infty} (-x^3)^\ell + x \sum_{\ell=0}^{\infty} (-x^3)^\ell, \tag{1.3}$$

so that

$$a_{s+3} = -a_s, \qquad s \in \mathbb{Z}_+, \tag{1.4a}$$

$$a_0 = 1, \qquad a_1 = 1, \qquad a_2 = 0 : \tag{1.4b}$$

the 6-periodicity results from 3-antiperiodicity.

This is so far unremarkable. However,

$$\frac{1}{1 - x + x^2} = \frac{1}{1 - x(1 - x)} = \sum_{m=0}^{\infty} x^m (1 - x)^m \tag{1.5a}$$

$$= \sum_{m \geq 0} x^m \sum_{k=0}^{m} \binom{m}{k} (-1)^k x^k = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k}. \tag{1.5b}$$

Thus, the binomial sums

$$s_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \tag{1.6a}$$

are equal to

$$1, 1, 0, -1, -1, 0 \qquad \text{for} \qquad n \equiv 0, 1, 2, 3, 4, 5 \pmod{6}. \tag{1.6b}$$

Such periodicity is very rare for binomial sums; see some examples and references in [12, 13]. The problem is: can this periodicity be quantized, in the sense of *q*-mathematics?

2 Guessing an answer

We need to find a suitable *q*-analog of the series $\sum_{m=0}^{\infty} x^m (1 - x)^m$ (1.5a). After some experimenting and rescaling, the following sum suggests itself for consideration:

$$S(x) = S(x; q) = \sum_{m=0}^{\infty} x^m (1 \dot{-} x)^m \tag{2.1a}$$

$$= \sum_{m=0}^{\infty} x^m (x; q)_m, \tag{2.1b}$$

where

$$(a \dot{+} b)^m = \left[\prod_{i=0}^{m} \left(a + q^i b \right) \right] / (a + q^m b) = \begin{cases} (a + b) \cdots \left(a + q^{m-1} b \right), & m \in \mathbb{N}, \\ 1, & m = 0 \end{cases} \tag{2.2}$$

and (see [2, p. 487])

$$(x; q)_m = \left[\prod_{i \geq 0}^{m} \left(1 - q^i x \right) \right] / (1 - q^m x) = \begin{cases} (1 - x) \cdots \left(1 - q^{m-1} x \right), & m \in \mathbb{N}, \\ 1, & m = 0. \end{cases} \tag{2.3}$$

By Euler's formula [2, p. 492],

$$(1 \dot{-} x)^m = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2}} (-x)^k, \tag{2.4}$$

where

$$\begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m] \cdots [m-k+1]}{[1] \cdots [k]}, \qquad k \in \mathbb{N}; \qquad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1, \tag{2.5}$$

are the q-binomial coefficients, and

$$[m] = [m]_q = (1 - q^m)/(1 - q) \tag{2.6}$$

is a q-analog of the classical number (or object) m.

Substituting (2.4) into (2.1), we get

$$S(x) = \sum_{m=0}^{\infty} x^m (1 \dot{-} x)^m = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k \\ k \end{bmatrix}_q q^{\binom{k}{2}}. \tag{2.7}$$

Thus, as a q-analog of the classical binomial sum (1.6a) we obtain

$$S_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k \\ k \end{bmatrix} q^{\binom{k}{2}}. \tag{2.8}$$

Is it any good? Calculating a few terms, we find:

$$S(x) = 1 + x + 0 \cdot x^2 - qx^3 - q^2 x^4 + 0 \cdot x^5 + q^5 x^6 + q^7 x^7 + 0 \cdot x^8 + O\left(x^9\right). \tag{2.9}$$

This looks promising, as the 3-antiperiodicity is preserved here, in q-clothes, but there are not enough terms to guess the rule for the q-exponents. We need some sort of a functional equation to determine that rule, assuming it exists.

We proceed as follows. Set

$$S(x, a) = S(x, a; q) = \sum_{m=0}^{\infty} a^m x^m (1 \dot{-} x)^m. \tag{2.10}$$

Since

$$(1 \dot{-} x)^{m+1} = (1 - x)(1 \dot{-} qx)^m, \tag{2.11}$$

we find that

$$S(x, a) = 1 + ax(1 - x) \sum_{m=0}^{\infty} (ax)^m (1 \dot{-} qx)^m = 1 + ax(1 - x)S(qx, a/q). \tag{2.12a}$$

Therefore, if we write

$$S(x, a) = \sum_{n=0}^{\infty} S_n(a) x^n, \tag{2.12b}$$

the functional equation (2.12a) converts itself into

$$S_{n+2}(a) = aS_{n+1}(a/q)q^{n+1} - aS_n(a/q)q^n, \qquad n \in \mathbb{Z}_+, \tag{2.13a}$$

$$S_1(a) = a, \qquad S_0(a) = 1. \tag{2.13b}$$

In particular, for $a = q$, we find:

$$S_0(q) = 1, \qquad S_1(q) = q, \tag{2.14a}$$

$$S_{n+2}(q) = q^{n+1}(qS_{n+1} - S_n), \qquad S_n = S_n(1), \qquad n \in \mathbb{Z}_+. \tag{2.14b}$$

Using the expansion (2.9), we obtain then

$$S(x,q) = \sum_{m=0}^{\infty} (qx)^m (1 \stackrel{.}{-} x)^m = 1 + qx + \left(q^2 - q\right)x^2 - q^2 x^3 - q^5 x^4$$
$$- \left(q^2 - q^7\right)x^5 + q^7 x^6 + q^{12} x^7 + \left(q^{15} - q^{12}\right)x^8 + O\left(x^9\right). \tag{2.15}$$

This is a bit less simple than (2.9) but still looks enticing. Moreover, comparing both expansions we can't fail to notice some coincidences and regularities; the following general ansatz suggests itself:

$$S_{6n}(q) = q^{x(n)}, \qquad\qquad S_{6n} = q^{x(n)-2n}, \tag{2.16a}$$

$$S_{6n+1}(q) = q^{y(n)}, \qquad\qquad S_{6n+1} = q^{x(n)}, \tag{2.16b}$$

$$S_{6n+2}(q) = q^{y(n)}\left(q^{2n+1} - 1\right), \qquad S_{6n+2} = 0, \tag{2.16c}$$

$$S_{6n+3}(q) = -q^{u(n)}, \qquad\qquad S_{6n+3} = -q^{u(n)-2n-1}, \tag{2.16d}$$

$$S_{6n+4}(q) = -q^{v(n)}, \qquad\qquad S_{6n+4} = -q^{u(n)}, \tag{2.16e}$$

$$S_{6n+5}(q) = -q^{v(n)}\left(q^{2n+2} - 1\right), \qquad S_{6n+5} = 0. \tag{2.16f}$$

Making this ansatz compatible with the relations (2.14) leads to the determination of all the unknown exponents $x(n)$, $y(n)$, $u(n)$, $v(n)$ in formulae (2.16):

$$y(n) = x(n) + 4n + 1, \tag{2.17a}$$

$$u(n) = x(n) + 6n + 2, \tag{2.17b}$$

$$v(n) = u(n) + 4n + 3, \tag{2.17c}$$

$$x(n+1) = 6n + 5 + u(n) \tag{2.17d}$$

$\Rightarrow$

$$x(n) = 6n^2 + n, \tag{2.18a}$$

$$y(n) = 6n^2 + 5n + 1, \tag{2.18b}$$

$$u(n) = 6n^2 + 7n + 2 \tag{2.18c}$$

$$v(n) = 6n^2 + 11n + 5. \tag{2.18d}$$

These are our *two* – conjectured so far – candidates for a quantum versions of the 6-periodic classical binomial sum (1.6). Are there any other candidates? It seems unlikely, at least if we insist on having relatively *compact* answers. For example,

$$S_4(a)/a^2 = q - a[3] + a^2, \tag{2.19a}$$

$$S_5(a)/a^3 = q[3] - a[4] + a^2, \tag{2.19b}$$

and for $a = q^L$ we don't get anything attractive unless $L = 0$ or 1.

Let us now collect our conjectured formulae into a series form. For $a = 1$, we get

$$S(x) = S(x, 1) = \sum_{n=0}^{\infty} S_n x^n$$

$$= \sum_{n=0}^{\infty} \left(x^{6n} q^{6n^2-n} + x^{6n+1} q^{6n^2+n} - x^{6n+3} q^{6n^2+5n+1} - x^{6n+4} q^{6n^2+7n+2} \right) \tag{2.20a}$$

$$= \sum_{n=0}^{\infty} \left\{ \left(x^{6n} q^{6n^2-n} - x^{6n+3} q^{6n^2+5n+1} \right) + \left(x^{6n+1} q^{6n^2+n} - x^{6n+4} q^{6n^2+7n+2} \right) \right\}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left\{ x^{3n} q^{n(3n-1)/2} + x^{3n+1} q^{n(3n+1)/2} \right\} : \tag{2.20b}$$

$$\sum_{m=0}^{\infty} x^m (1 \overset{.}{-} x)^m = \sum_{n=0}^{\infty} (-x)^{3n} q^{n(3n-1)/2} (1 + xq^n) . \tag{2.21}$$

Similarly, for $a = q$ we obtain

$$S(x, q) = \sum_{n=0}^{\infty} S_n(q) x^n$$

$$= \sum_{n=0}^{\infty} x^{6n} \left\{ q^{6n^2+n} + xq^{6n^2+5n+1} + x^2 q^{6n^2+5n+1} \left(q^{2n+1} - 1 \right) \right.$$

$$\left. - x^3 q^{6n^2+7n+2} - x^4 q^{6n^2+11n+5} - x^5 q^{6n^2+11+5} \left(q^{2n+2} - 1 \right) \right\} \tag{2.22a}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left\{ x^{3n} q^{n(3n+1)/2} + x^{3n+1} q^{n(3n+5)2} + x^{3n+2} q^{n(3n+5)/2} \left(q^{n+1} - 1 \right) \right\} : \tag{2.22b}$$

$$\sum_{m=0}^{\infty} q^m x^m (1 \overset{.}{-} x)^m = \sum_{n=0}^{\infty} (-x)^{3n} q^{n(3n+1)/2} \left\{ 1 + xq^{2n} + x^2 q^{2n} \left(q^{n+1} - 1 \right) \right\} . \tag{2.23}$$

In view of the functional equation (2.12), formula (2.21) implies formula (2.23). In the next Section we shall prove the former formula.

Remark 2.24. Formula (2.12a) shows that the parameter

$$y = ax \tag{2.25}$$

remains invariant under iteration. Therefore, let us set

$$R(x, y) = R(x, y; q) = \sum_{m=0}^{\infty} y^m (1 \overset{.}{-} x)^m . \tag{2.26}$$

Then

$$R(x, y) = 1 + y(1 - x) R(qx, y) . \tag{2.27}$$

Setting

$$R(x, y) = \sum_{\ell=0}^{\infty} c_\ell(y) x^\ell , \tag{2.28}$$

we convert the functional equation (2.27) into

$$c_0 = 1 + yc_0, \tag{2.29a}$$

$$c_{\ell+1} = yq^{\ell+1}c_{\ell+1} - yq^{\ell}c_{\ell} \qquad \Rightarrow \tag{2.29b}$$

$$c_{\ell} = \frac{(-y)^{\ell}q^{\binom{\ell}{2}}}{(1 \mathbin{\dot-} y)^{\ell+1}}, \qquad \ell \in \mathbb{Z}_+ \qquad \Rightarrow \tag{2.30}$$

$$R(x,y) = \sum_{m=0}^{\infty} y^m (1 \mathbin{\dot-} x)^m = \sum_{\ell=0}^{\infty} \frac{(-xy)^{\ell}}{(1 \mathbin{\dot-} y)^{\ell+1}} q^{\binom{\ell}{2}} \qquad \Rightarrow \tag{2.31}$$

$$S(x,a) = \sum_{m=0}^{\infty} (ax)^m (1 \mathbin{\dot-} x)^m = \sum_{\ell=0}^{\infty} \frac{(-ax^2)^{\ell}}{(1 \mathbin{\dot-} ax)^{\ell+1}} q^{\binom{\ell}{2}}. \tag{2.32}$$

Neither of these expansions, however, is helpful for out task of proving formula (2.21). We need something completely different. Notice that for $q = 1$, formula (2.31) yields:

$$\frac{1}{1 - y(1-x)} = \sum_{m=0}^{\infty} y^m (1-x)^m = \sum_{\ell=0}^{\infty} \frac{(-xy)^{\ell}}{(1-y)^{\ell+1}} = \frac{1}{1-y} \frac{1}{1 + \dfrac{xy}{1-y}}. \tag{2.33}$$

3 Euler's tower

We have to prove formula (2.21). In the form (2.20b), it is:

$$\sum_{m=0}^{\infty} x^m (1 \mathbin{\dot-} x)^m = \sum_{n=0}^{\infty} (-1)^n \left\{ x^{3n} q^{n(3n-1)/2} + x^{3n+1} q^{n(3n+1)/2} \right\}. \tag{3.1}$$

This is an identity between formal power series in x, with coefficients that are polynomials in q. We can not therefore let x to be 1 (unlike q). But let's get wild for a moment. The RHS of formula (3.1) for $x = 1$ is:

$$\sum_{n=0}^{\infty} (-1)^n \left\{ q^{n(3n-1)/2} + q^{n(3n+1)/2} \right\} = 1 + \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{3.2}$$

This is essentially the RHS of Euler's famous identity, conjectured by him in 1741:

$$\prod_{j=1}^{\infty} \left(1 - q^j\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{3.3}$$

(See [15, Ch. 6] for an English translation of Euler's fascinating memoir.) Nowadays, Euler's formula (3.3) is subsumed by the more general Jacobi triple product identity

$$\prod_{n=1}^{\infty} \left(1 + Q^{2n-1}z\right) \left(1 + Q^{2n-1}z^{-1}\right) \left(1 - Q^{2n}\right) = \sum_{n=-\infty}^{\infty} Q^{n^2} z^n \tag{3.4}$$

(see [16, p. 10 and p. 186]). Euler's formula (3.3) results from (3.4) when one makes the substitution

$$Q = q^{3/2}, \qquad z = -q^{1/2}. \tag{3.5}$$

Neither Euler's formula (3.3) nor Jacobi's formula (3.4) seem of any help to our identity (3.1). However, Euler's ingenious *Proof* of his identity (see [24, p. 281]) has enough ingredients in it to establish (3.1). Weil describes Euler's Proof as "another dazzling display of algebraic virtuosity but quite elementary ...".

Euler sets

$$P_0 = \prod_{j=1}^{\infty} \left(1 - q^j\right) \tag{3.6}$$

and notices that

$$P_0 = 1 - q - q^2 P_1, \tag{3.7}$$

where

$$P_1 = \sum_{\nu=0}^{\infty} q^{\nu}(1 - q) \cdots \left(1 - q^{\nu+1}\right). \tag{3.8}$$

This follows from the easily proven by induction identity

$$\prod_{j=1}^{n}(1 - \alpha_j) = 1 - \alpha_1 - \sum_{k=2}^{n} \alpha_k(1 - \alpha_1) \cdots (1 - \alpha_{k-1}), \qquad n \geq 2, \tag{3.9}$$

by setting $\alpha_j = q^j$, replacing k by $\nu + 2$, and then passing to the limit $n \to \infty$. Euler then introduces the infinite tower of series P_n:

$$P_n = \sum_{\nu=0}^{\infty} q^{\nu n} \left(1 - q^n\right) \cdots \left(1 - q^{n+\nu}\right), \qquad n \in \mathbb{N}, \tag{3.10}$$

and shows by a *different argument* that

$$P_n = 1 - q^{2n+1} - q^{3n+2} P_{n+1}, \qquad n \in \mathbb{N}. \tag{3.11}$$

Euler's argument for $n > 0$ is ingenious:

$$
\begin{aligned}
P_n &= \sum_{\nu=0}^{\infty} q^{\nu n} \left(1 - q^n\right) \cdots \left(1 - q^{n+\nu}\right) \\
&= (1 - q^n) + q^n \left(1 - q^n\right) \left(1 - q^{n+1}\right) + \sum_{\mu=2}^{\infty} q^{\mu n} \left(1 - q^n\right) \cdots \left(1 - q^{n+\mu}\right) \\
&= 1 - q^n + q^n \left(1 - q^{n+1}\right) - q^{2n} \left(1 - q^{n+1}\right) \\
&\quad + \sum_{\mu=2}^{\infty} q^{\mu n} \left(1 - q^{n+1}\right) \cdots \left(1 - q^{n+\mu}\right) - q^n \sum_{\mu=2}^{\infty} q^{\mu n} \left(1 - q^{n+1}\right) \cdots \left(1 - q^{n+\mu}\right)
\end{aligned}
$$

$$= 1 - q^{2n+1} - q^{2n}\left(1 - q^{n+1}\right) + \sum_{\nu=0}^{\infty} q^{\nu n + 2n}\left(1 - q^{n+1}\right) \cdots \left(1 - q^{n+2+\nu}\right)$$

$$- \sum_{\mu=1}^{\infty} q^{(\mu+1)n}\left(1 - q^{n+1}\right) \cdots \left(1 - q^{n+\mu}\right) + q^{2n}\left(1 - q^{n+1}\right)$$

$$= 1 - q^{2n+1} + \sum_{\nu=0}^{\infty} q^{\nu(n+1)+2n-\nu}\left(1 - q^{n+1}\right) \cdots \left(1 - q^{n+1+\nu}\right)\left(1 - q^{n+2+\nu}\right)$$

$$- \sum_{\nu=0}^{\infty} q^{(\nu+2)n}\left(1 - q^{n+1}\right) \cdots \left(1 - q^{n+1+\nu}\right)$$

$$= 1 - q^{2n+1} + \sum_{\nu=0}^{\infty} q^{\nu(n+1)}\left(1 - q^{n+1}\right) \cdots \left(1 - q^{n+1+\nu}\right)\left\{ q^{2n-\nu}\left(1 - q^{n+2+\nu}\right) - q^{2n-\nu}\right\}$$

$$= 1 - q^{2n+1} - q^{3n+2}\sum_{\nu=0}^{\infty} q^{\nu(n+1)}\left(1 - q^{n+1}\right) \cdots \left(1 - q^{n+1+\nu}\right) = 1 - q^{2n+1} - q^{3n+2}P_{n+1}.$$

Collecting all the relations (3.11) together, we find

$$\prod_{j=1}^{\infty}\left(1 - q^j\right) = P_0 = 1 - q - q^2 P_1 = 1 - q - q^2\left(1 - q^3 - q^5 P_2\right)$$

$$= 1 - q - q^2\left(1 - q^3\right) + q^{2+5}\left(1 - q^5 - q^8 P_3\right) = \cdots$$

$$= 1 - q + \sum_{n=1}^{\infty}(-1)^n q^{2+5+\cdots+(3n-1)}\left(1 - q^{2n+1}\right) \tag{3.12}$$

$$= 1 - q + \sum_{n=1}^{\infty}(-1)^n q^{n(3n+1)/2}\left(1 - q^{2n+1}\right) = \sum_{n=0}^{\infty}(-1)^n q^{n(3n+1)/2}\left(1 - q^{2n+1}\right) \tag{3.13}$$

$$= \sum_{n=0}^{\infty}(-1)^n q^{n(3n+1)/2} + \sum_{n=0}^{\infty}(-1)^{n+1} q^{(3n^2+n+4n+2)/2} = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}. \tag{3.14}$$

While we are at it, let's notice that a similar iteration yields

$$P_\ell = \sum_{n=0}^{\infty}(-1)^n q^{3\ell n + n(3n+1)/2}(1 - q^{2\ell+2n+1}), \qquad \ell \in \mathbb{Z}_+. \tag{3.15}$$

From this, we can readily see that

$$\bar{P}_\ell := 1 + q^\ell P_\ell = \sum_{n=0}^{\infty}(-1)^n\left\{ q^{3\ell n} q^{n(3n-1)/2} + q^{(3n+1)\ell} q^{n(3n+1)/2}\right\}. \tag{3.16}$$

It now only remains to compare $\bar{P}_\ell$ with

$$\bar{S}_\ell = S(q^\ell; q) = S(x; q)|_{x=q^\ell}, \qquad \forall\, \ell \in \mathbb{N}: \tag{3.17}$$

$$\bar{S}_\ell = S(q^\ell; q) = \sum_{m=0}^{\infty} q^{\ell m}(1 \dot{-} q^\ell)^m = 1 + \sum_{m=1}^{\infty} q^{\ell m}\left(1 - q^\ell\right) \cdots \left(1 - q^{\ell+m-1}\right), \tag{3.18a}$$

$$\bar{P}_\ell = 1 + q^\ell \sum_{\nu=0}^{\infty} q^{\nu \ell}\left(1 - q^\ell\right) \cdots \left(1 - q^{\ell+\nu}\right), \tag{3.18b}$$

and these are identical. Thus,

$$\bar{S}_\ell = \bar{P}_\ell, \qquad \ell \in \mathbb{N}. \tag{3.19}$$

Comparing formulae (3.1) and (3.16), we get

$$\left\{ \sum_{m=0}^{\infty} x^m (1 \dot{-} x)^m \right\}\Bigg|_{x=q^\ell}$$

$$= \left(\sum_{n=0}^{\infty} (-1)^n \left\{ x^{3n} q^{n(3n-1)/2} + x^{3n+1} q^{n(3n+1)/2} \right\} \right)\Bigg|_{x=q^\ell}, \qquad \forall\, \ell \in \mathbb{N}. \tag{3.20}$$

This proves formula (3.1), because:

Lemma 3.21. *Suppose $f \in \mathbb{C}[[x,y]]$ is formal power series in x, y, and let $r(1)$, $r(2)$, $\ldots$ be an increasing sequence of positive intergers. If*

$$f\left(y^{r(i)}, y\right) = f(x,y)\big|_{x=y^{r(i)}} = 0, \qquad \forall\, i \in \mathbb{N}, \tag{3.22}$$

then f is identically zero.

Proof. If $f(y^{r(1)}, y) = 0$ then f is divisible by $x - y^{r(1)}$:

$$f(x,y) = \left(x - y^{r(1)}\right) f_1(x,y). \tag{3.23}$$

Continuing on, we find that

$$f(x,y) = \left\{ \prod_{i=1}^{k} \left(x - y^{r(i)}\right) \right\} f_k(x,y) \tag{3.24}$$

Thus, all homogeneous components of f of degrees $< k$ vanish, and k is arbitrary. ∎

4   Euler's tower revisited

> *Woman:* Your're a pugilist, arent't you?
> *World champ boxer Rocky Graziano:* Nah,
> I'm just a prizefighter.

The Euler relation (3.11)

$$P_n = 1 - q^{2n+1} - q^{3n+2} P_{n+1}, \qquad n \in \mathbb{N}, \tag{4.1}$$

for the sequence of series

$$P_n = \sum_{\nu=0}^{\infty} q^{\nu n} \left(1 - q^n\right) \cdots \left(1 - q^{n+\nu}\right), \qquad n \in \mathbb{N}, \tag{4.2}$$

has an implicit gem hidden inside. To make it explicit, notice that in the definition of P_n (4.2) the index n could be treated as a formal parameter and not necessarily as a positive integer. So, set

$$q^n = x \tag{4.3}$$

and define

$$\mathcal{P}(x) = \mathcal{P}(x;q) = \sum_{\nu=0}^{\infty} x^\nu (1-x) \cdots (1 - xq^\nu). \tag{4.4}$$

Applying the Euler argument to $\mathcal{P}(x)$, we get:

$$\mathcal{P}(x) = \sum_{\nu=0}^{\infty} x^\nu (1-x) \cdots (1 - xq^\nu) = (1-x) + x(1-x)(1-xq)$$

$$+ \sum_{\mu=2}^{\infty} x^\mu (1-x) \cdots (1 - xq^\mu) = (1-x) + x(1-xq) - x^2(1-xq)$$

$$+ \sum_{\mu=2}^{\infty} x^\mu (1-xq) \cdots (1-xq^\mu) - x \sum_{\mu=2}^{\infty} x^\mu (1-xq) \cdots (1-xq^\mu)$$

$$= 1 - qx^2 - x^2(1-xq) + \sum_{\nu=0}^{\infty} x^{\nu+2} (1-xq) \cdots \left(1 - xq^{\nu+1}\right)\left(1 - xq^{\nu+2}\right)$$

$$- \sum_{\mu=1}^{\infty} x^{\mu+1} (1-xq) \cdots (1-xq^\mu) + x^2(1-xq)$$

$$= 1 - qx^2 + x^2 \sum_{\nu=0}^{\infty} x^\nu (1-xq) \cdots \left(1 - xq^{\nu+1}\right) \left\{\left(1 - xq^{\nu+2}\right) - 1\right\}$$

$$= 1 - qx^2 - x^3 q^2 \sum_{\nu=0}^{\infty} (xq)^\nu (1-xq) \cdots (1 - xq \cdot q)^\nu = 1 - qx^2 - q^2 x^3 \mathcal{P}(xq) : \tag{4.5}$$

$$\mathcal{P}(x;q) = 1 - qx^2 - q^2 x^3 \mathcal{P}(xq;q). \tag{4.6}$$

Writing

$$\mathcal{P}(x;q) = \sum_{\ell=0}^{\infty} c_\ell(q) x^\ell, \tag{4.7}$$

we can transform the relation (4.6) into

$$c_{\ell+3} = -q^{\ell+2} c_\ell, \tag{4.8a}$$

$$c_0 = 1, \qquad c_1 = 0, \qquad c_2 = -q \qquad \Rightarrow \tag{4.8b}$$

$$c_{3\ell} = (-1)^\ell q^{\ell(3\ell+1)/2}, \tag{4.9a}$$

$$c_{3\ell+1} = 0, \tag{4.9b}$$

$$c_{3\ell-1} = (-1)^\ell q^{\ell(3\ell-1)/2} \qquad \Rightarrow \tag{4.9c}$$

$$\mathcal{P}(x) = \sum_{\nu=0}^{\infty} x^\nu (1-x) \cdots (1 - xq^\nu) \tag{4.10a}$$

$$= 1 + \sum_{\ell=1}^{\infty} (-1)^\ell \left\{ q^{\ell(3\ell-1)/2} x^{3\ell-1} + q^{\ell(3\ell+1)/2} x^{3\ell} \right\}. \tag{4.10b}$$

If we notice that

$$S(x) = 1 + x\mathcal{P}(x), \tag{4.11}$$

then from formula (4.10b) we obtain at once that

$$S(x) = \sum_{m=0}^{\infty} x^m (1 \stackrel{.}{-} x)^m \tag{4.12a}$$

$$= \sum_{\ell=0}^{\infty} (-1)^\ell \left\{ x^{3\ell} q^{\ell(3\ell-1)/2} + x^{3\ell+1} q^{\ell(3\ell+1)/2} \right\}, \tag{4.12b}$$

which is our conjectured formula (2.20b). The functional equation (4.6) in the S-language becomes:

$$S(x) = 1 + x - qx^3 S(qx); \tag{4.13}$$

this equation by itself implies formula (4.12b) at once.

Remark 4.14. The identity (4.10) is essentially due to Euler, but it was repeatedly rediscovered many times since. See [1, p. 282] for more on that.

Remark 4.15. What is the orgin of number 6, and are there interesting periodic sequences of period other than 6? I don't yet have a definite answer to these questions but suspect that our 6-periodic sequence is a camouflaged group of units of the field $\mathbb{Q}[\sqrt{-3}]$; this suggests that there exists a 4-periodic sequence attached to the units ± 1, $\pm i$ of $\mathbb{Q}[\sqrt{-1}]$, and if a number field $K \supset \mathbb{Q}$ has a finite group of units then these can be organized into a periodic (vector) sequence.

5   Recurrence relations of second order

The space of 3-antiperiodic sequences is 3-dimensional. It contains a special 2-dimensional subspace consisting of 2nd-order recurrent sequences $\{u_n\}$ satisfying the relation

$$u_{n+2} = u_{n+1} - u_n. \tag{5.1}$$

Indeed, for $u_0 = a$, $u_1 = b$, the sequence u_n starts as

$$a, \ b, \ b - a, \ -a, \ -b, \ -b + a \mid a, \ b, \ \ldots \tag{5.2}$$

(In the language of characteristic polynomials: $\lambda^3 + 1$ is divisible by $\lambda^2 - \lambda + 1$).

Our sequence of binomial sums s_n (1.6a)

$$s_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \tag{5.3}$$

is of this special form, with $a = b = 1$. This is true a posteriori, from comparing formulae (5.2) and (1.6b), but can be easily seen directly from the definition of s_n (5.3):

$$s_{n+2} - s_{n+1} = \sum_{k \geq 0} (-1)^k \left\{ \binom{n+2-k}{k} - \binom{n+1-k}{k} \right\} = \sum (-1)^k \binom{n+1-k}{k-1}$$

$$= \sum (-1)^{k+1} \binom{n-k}{k} = -s_n. \tag{5.4}$$

In quantizing s_n in § 2, we went along the route of generating functions, quantizing the latter for the sequence s_n:

$$\sum_{n=0}^{\infty} s_n x^n = \sum_{m=0}^{\infty} x^m (1-x)^m. \tag{5.5}$$

Generating functions make a very useful and powerful device (see [25].) However, when one attempts to quantize a classical object, property, relation, etc., the generating function methods may unnecessarily restrict one's choice. A cautionary tale follows.

Let's try to quantize the *calculation* (5.4). Set

$$f_n = f_{n|L} = f_{n|L}(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(-q^L\right)^k q^{\theta(n,k)} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \tag{5.6}$$

with some unknown function $\theta(n,k)$ to be specified later on. We shall use the formulae

$$\begin{bmatrix} \alpha + 1 \\ r \end{bmatrix} = q^r \begin{bmatrix} \alpha \\ r \end{bmatrix} + \begin{bmatrix} \alpha \\ r-1 \end{bmatrix} \tag{5.7a}$$

$$= \begin{bmatrix} \alpha \\ r \end{bmatrix} + q^{\alpha+1-r} \begin{bmatrix} \alpha \\ r-1 \end{bmatrix}. \tag{5.7b}$$

First, by formula (5.7a)

$$f_{n+2|L} = \sum_k \left(-q^L\right)^k q^{\theta(n+2,k)} \begin{bmatrix} n+2-k \\ k \end{bmatrix}$$

$$= \sum_k \left(-q^L\right)^k q^{\theta(n+2,k)} \left\{ q^k \begin{bmatrix} n+1-k \\ k \end{bmatrix} + \begin{bmatrix} n+1-k \\ k-1 \end{bmatrix} \right\}$$

$$= \sum_k \left(-q^{L+1}\right)^k q^{\theta(n+2,k)} \begin{bmatrix} n+1-k \\ k \end{bmatrix} - q^L \sum_k \left(-q^L\right)^k q^{\theta(n+2,k+1)} \begin{bmatrix} n-k \\ k \end{bmatrix}. \tag{5.8}$$

This is an opaque expression. If we simplify it by having $L = 0$, demanding that

$$\theta(n+2,k) + k = \theta(n+1,k) \tag{5.9}$$

modulo rescaling of f_n by a function of n, and then requiring that

$$\{\theta(n+2,k+1) - \theta(n,k)\} \quad \text{is } k\text{-independent}, \tag{5.10}$$

we find that

$$\theta(n,k) = k(k-n), \tag{5.11}$$

and formula (5.8) yields:

$$f_{n+2} = f_{n+1} - q^{-(n+1)} f_n. \tag{5.12}$$

Changing q into q^{-1}, we get:

$$f_{n+2} = f_{n+1} - q^{n+1} f_n, \qquad f_0 = f_1 = 1, \tag{5.13}$$

$$f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \tag{5.14}$$

because

$$[n]!_{q^{-1}} = [n]!q^{-\binom{n}{2}},$$
(5.15a)

$$\begin{bmatrix} n \\ r \end{bmatrix}_{q^{-1}} = \begin{bmatrix} n \\ r \end{bmatrix}_q q^{r(r-n)}.$$
(5.15b)

The sequence f_n is:

$$1, \ 1, \ 1 - q, \ 1 - q - q^2, \ 1 - q - q^2 - q^3 + q^4, \ 1 - q - q^2 - q^3 + q^5 + q^6,$$
$$1 - q - q^2 - q^3 + 2q^6 + q^7 + q^8 + q^9, \ \ldots$$
(5.16a)

and it doesn't appear interesting; however, its limit f_∞ certainly is: from formula (5.14) with $n = \infty$, we get

$$f_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2}}{(1 \dot{-} q)^k}.$$
(5.16b)

Secondly, if we use formula (5.7b) instead of (5.7a), we find (with $L = 0$):

$$f_{n+2} = \sum (-1)^k q^{\theta(n+2,k)} \left\{ \begin{bmatrix} n+1-k \\ k \end{bmatrix} + q^{n+2-2k} \begin{bmatrix} n+1-k \\ k-1 \end{bmatrix} \right\}$$
$$= \sum (-1)^k q^{\theta(n+2,k)} \begin{bmatrix} n+1-k \\ k \end{bmatrix} - q^n \sum (-1)^k q^{\theta(n+2,k+1)-2k} \begin{bmatrix} n-k \\ k \end{bmatrix}.$$
(5.17)

The first summand in (5.17) implies that $\theta(n, k)$ is n-independent, and the second summand yields

$$\theta(n, k) = k^2.$$
(5.18)

Thus,

$$f_{n+2} = f_{n+1} - q^{n+1} f_n, \qquad f_0 = f_1 = 1,$$
(5.19)

$$f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q,$$
(5.20)

which is the same as formula (5.14).

If we set

$$F(x) = \sum_{n=0}^{\infty} f_n x^n,$$
(5.21)

then formula (5.19) converts itself into

$$F(x)(1-x) = 1 - qx^2 F(qx),$$
(5.22)

whence, by repeated iteration,

$$F(x) = \sum_{k=0}^{\infty} \frac{\left(-x^2\right)^k q^{k^2}}{(1 \dot{-} x)^{k+1}}.$$
(5.23)

Our original sequence S_n (2.8),

$$S_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k \\ k \end{bmatrix} q^{\binom{k}{2}} \tag{5.24}$$

satisfies, as can be directly checked, the following q-analog of the relation (5.1):

$$S_n = q^{\lfloor n/3 \rfloor} S_{n-1} - q^{\lfloor 2n/3 \rfloor - 1} S_{n-2}; \tag{5.25}$$

the more general sequence $S_n(a)$ (2.12):

$$S_n(a) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k \\ k \end{bmatrix} q^{\binom{k}{2}} a^{n-k}, \tag{5.26}$$

for $a = q$ satisfies the easily verified relation

$$S_n(q) = q^{\lfloor (n+2)/3 \rfloor} S_{n-1}(q) - q^{\lfloor (2n+1)/3 \rfloor} S_{n-2}(q). \tag{5.27}$$

6   Chebyshev polynomials and series

The special 3-antiperiodic relation (5.1)

$$u_{n+2} = u_{n+1} - u_n \tag{6.1}$$

is satisfied by the values of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ at $x = 1/2$. Let us recall the salient facts.

Originally, Chebyshev polynomials appeared as the least deviant monic polynomials of a fixed degree, n say, on the interval $-1 \leq x \leq 1$. For the norm

$$\|f\| = \max_{-1 \leq x \leq 1} |f(x)|,$$

the answer is $T_n(x)/2^{n-1}$,

$$T_n(x) = \cos(n\theta), \qquad x = \cos\theta, \tag{6.2}$$

being the Chebyshev polynomial of the 1st kind; for the norm

$$\|f\| = \int_{-1}^{1} |f(x)| dx,$$

the answer is $U_n(x)/2^n$,

$$U_n(x) = \frac{\sin((n+1)\theta))}{\sin\theta}, \qquad x = \cos\theta, \tag{6.3}$$

being the Chebyshev polynomial of the 2nd kind.

The following table is extracted from [11, p. 287]:

$$T_0(x) = 1,$$
$$T_1(x) = x,$$
$$T_2(x) = 2x^2 - 1,$$
$$T_3(x) = 4x^3 - 3x,$$
$$T_4(x) = 8x^4 - 8x^2 + 1,$$
$$T_5(x) = 16x^5 - 20x^3 + 5x,$$
$$T_6(x) = 32x^6 - 48^4 + 18x^2 - 1,$$
$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x,$$
$$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1,$$
$$T_9(x) = 256x^9 - 576x^7 + 232x^5 - 120^3 + 9x; \tag{6.4}$$
$$U_0(x) = 1,$$
$$U_1(x) = 2x,$$
$$U_2(x) = 4x^2 - 1,$$
$$U_3(x) = 8x^3 - 4x,$$
$$U_4(x) = 16x^4 - 12x^2 + 1,$$
$$U_5(x) = 32x^5 - 32x^3 + 6x,$$
$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1,$$
$$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x,$$
$$U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1,$$
$$U_9(x) = 512x^9 - 1024x^7 + 672x^5 - 106x^3 + 10x. \tag{6.5}$$

The table (6.4) suggests that

$$T_p(x) \equiv x^p \pmod{p}, \qquad p \text{ an odd prime}, \tag{6.6}$$

which is true because of the easily verifiable formula

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k} \left(x^2 - 1\right)^k, \tag{6.7}$$

and this interesting congruence is listed as an exercise in [2, p. 117]; there are more exercises on that page, about other interesting arithmetic properties of the Chebyshev polynomials of the 1st and 2nd kind. Some others properties not on that page are:

$$T_{p^2}(x) \equiv T_p\left(x^p\right) \pmod{p^2}, \qquad p \text{ an odd prime}, \tag{6.8a}$$
$$T_{2^{n+1}}(x) \equiv 1 \pmod{2^{2n+1}}, \qquad n \in \mathbb{Z}_+, \tag{6.8b}$$

which follow from formula (6.6) and formula (6.17) below. No doubt there exist many more interesting arithmetic properties of the Chebyshev polynomials.

Such as. Define the sequence $\{\gamma_n\}$ by the rule:

$$\gamma_{n+2} = 2\gamma_{n+1} + \gamma_n, \qquad \gamma_0 = 0, \qquad \gamma_1 = 1. \tag{6.9a}$$

Then

$$T_{2n}\left(\pm\sqrt{-1}\right) = 1 + (-1)^n\gamma_n^2, \qquad n \in \mathbb{Z}_+, \tag{6.9b}$$

$$T_{2n+1}\left(\pm\sqrt{-1}\right) = \pm\sqrt{-1}\left\{-1 + (-1)^n\left(\gamma_{n+1}^2 - \gamma_n^2\right)/2\right\}, \qquad n \in \mathbb{Z}_+. \tag{6.9c}$$

Most trigonometric formulae translate into formulae for the Chebyshev polynomials in view of the representations (6.2) and (6.3). For example, formula

$$\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2},$$

for $u = (n+2)\theta$, $v = n\theta$; yields

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x); \tag{6.10a}$$

formula

$$\sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$$

yields

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \tag{6.10b}$$

for $u = (n+2)\theta$, $v = n\theta$; and

$$U_n(x) - U_{n-2}(x) = 2T_n(x) \tag{6.11}$$

for $u = (n+1)\theta$, $v = -(n-1)\theta$. (Many more formulae involving Chebyshev polynomials can be found in [3, § 10.11], [17, 7], and [14, § 5.7].)

Formulae (6.10) for $x = 1/2$ show that $\{T_n(1/2)\}$ and $\{U_n(1/2)\}$ are 3-antiperiodic sequences of the form (6.1), with

$$T_0(1/2) = 1, \qquad T_1(1/2) = 1/2, \tag{6.12a}$$

$$U_0(1/2) = 1, \qquad U_1(1/2) = 1. \tag{6.12b}$$

The preceding Sections thus have dealt with quantum aspects of Chebyshev polynomials $U_n(x)$'s at just *one point* $x = 1/2$. We won't attempt anything quantum in this Section, as this would be a rather formidable undertaking that is better left to the interested reader. Instead, we shall explore the problem of *interpolating* the function $u(n) = u_n$ from integers to real and complex numbers.

We need a few more standard formulae.

Set temporarily $\tilde{U}_n(x) = (n+1)^{-1}dT_{n+1}/dx$. Differentiating formula (6.10a), we get:

$$(n+1)(\tilde{U}_{n+1} - 2x\tilde{U}_n + \tilde{U}_{n-1}) + (\tilde{U}_{n+1} - \tilde{U}_{n-1} - 2T_n) = 0. \tag{6.13}$$

This equation defines $\tilde{U}_n$'s recursively. From the table (6.4),

$$\tilde{U}_0 = T_1' = 1 = U_0, \tag{6.14a}$$

$$\tilde{U}_1 = T_2'/2 = 2x = U_1, \tag{6.14b}$$

and the U_n's also satisfy the equation (6.13) by formulae (6.10b) and (6.11). Therefore, $\tilde{U}_n = U_n$:

$$U_n(x) = \frac{1}{n+1} \frac{dT_{n+1}}{dx}. \tag{6.15}$$

(Alternatively, one gets formula (6.15) by differentiating formula (6.2) with respect to x.)
Formulae (6.10) and (6.15) yield:

$$T_n(1) = 1, \tag{6.16a}$$

$$U_n(1) = n + 1, \tag{6.16b}$$

$$T_n'(1) = n^2. \tag{6.16c}$$

Formula (6.2) implies that ([19, p. 45], [21, 27])

$$T_n(T_m(x)) = T_{nm}(x), \qquad n, m \in \mathbb{N}. \tag{6.17}$$

Thus, the polynomials T_n's form a commutative semigroup; below we shall see that this semigroup is in fact a discrete part of a commutative 1-dimensional *group* considered as either a formal group or as a group of holomorphic automorphisms of a neighborhood of the complex plane around $z = 1$. Formula (6.15) implies that $\{T_n$'s$\}$ are a more fundamental object than $\{U_n$'s$\}$.
Now, from the definition (6.2),

$$\frac{d^2 T_n}{d\theta^2} = -n^2 T_n. \tag{6.18a}$$

Since

$$\frac{d}{d\theta} = -\sin\theta \frac{d}{dx}, \qquad \frac{d^2}{d\theta^2} = \left(1 - x^2\right) \frac{d^2}{dx^2} - x \frac{d}{dx}, \tag{6.18b}$$

we get the classical equation

$$\left(1 - x^2\right) T_n'' - x T_n' + n^2 T_n = 0. \tag{6.19}$$

The T_n is the unique regular solution of the equation

$$\left(1 - x^2\right) y'' - x y' + n^2 y = 0 \tag{6.20a}$$

satisfying the initial condition

$$y(1) = 1. \tag{6.20b}$$

In the variable

$$z = \frac{1 - x}{2}, \tag{6.21}$$

the equation (6.20a) takes the hypergeometric form

$$z(1-z)\frac{d^2y}{dz^2} + \left(\frac{1}{2} - z\right)\frac{dy}{dz} + n^2y = 0, \tag{6.22}$$

and we recover another classical result:

$$T_n(x) = F\left(-n, n, \frac{1}{2}; \frac{1-x}{2}\right). \tag{6.23}$$

(The point $x = \frac{1}{2}$ directs us to the continuous 3-antiperiodic family $u(\alpha) = F\left(-\alpha, \alpha, \frac{1}{2}; \frac{1}{4}\right)$.)
  Formulae (6.4), (6.5), (6.10) imply that

$$T_n(0) = U_n(0) = 0, \qquad n \text{ odd}; \tag{6.24a}$$
$$T_{2n}(0) = U_{2n}(0) = (-1)^n. \tag{6.24b}$$

The differential equation (6.19) then yields

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!}(2x)^{n-2k}, \qquad n \in \mathbb{N}, \tag{6.25}$$

and then formula (6.15) yields

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k}(2x)^{n-2k}, \qquad n \in \mathbb{Z}_+; \tag{6.26}$$

another classical pair of very useful formulae.
  We now extend the index n in $T_n(x)$ by allowing n to be an arbitrary complex number or a formal parameter: we set

$$T_\alpha(x) = \sum_{k=0}^{\infty} c_{\alpha|k}(x-1)^k, \qquad c_{\alpha|0} = 1, \tag{6.27}$$

and require T_α to satisfy the differential equation

$$\left(1 - x^2\right)T_\alpha'' - xT_\alpha' + \alpha^2 T_\alpha = 0, \tag{6.28}$$

together with the boundary condition

$$T_\alpha(1) = 1. \tag{6.29}$$

This is equivalent to the relations

$$c_{\alpha|k+1} = \frac{\alpha^2 - k^2}{(k+1)(2k+1)}c_{\alpha|k}, \qquad k \in \mathbb{Z}_+ \quad \Rightarrow \tag{6.30}$$

$$c_{\alpha|k+1} = \frac{1}{(k+1)!(2k+1)!!}\prod_{i=0}^{k}\left(\alpha^2 - i^2\right), \qquad c_{\alpha|0} = 1. \tag{6.31}$$

Since

$$\lim_{k \to \infty} |c_{\alpha|k+1}/c_{\alpha|k}| = 1/2, \tag{6.32}$$

the series $T_\alpha(x)$ (6.27) converges for

$$|x - 1| < 2. \tag{6.33}$$

The series terminates only when α is an integer. It is readily verified that

$$T_{\alpha+2} - 2(1 + (x-1))T_{\alpha+1} + T_\alpha = 0, \qquad \forall\, \alpha, \tag{6.34}$$

and that

$$U_\alpha - U_{\alpha-2} = 2T_\alpha, \qquad \forall\, \alpha, \tag{6.35}$$

where

$$U_\alpha(x) = \frac{1}{\alpha+1}\frac{dT_\alpha}{dx}, \qquad \forall\, \alpha. \tag{6.36}$$

From this it follows that

$$U_{\alpha+2} - 2((1 + (x-1))U_{\alpha+1} + U_\alpha = 0, \qquad \forall\, \alpha. \tag{6.37}$$

Also, formula (6.30) shows that

$$T_{-\alpha} = T_\alpha, \qquad \forall\, \alpha. \tag{6.38}$$

Finally,

$$T_\alpha(T_\beta(x)) = T_{\alpha\beta}(x), \qquad \forall\, \alpha, \beta. \tag{6.39a}$$

Indeed, the difference

$$T_\alpha(T_\beta(x)) - T_{\alpha\beta}(x),$$

for each fixed degree k of $(x-1)^k$, is a *polynomial* in α, β which vanishes, by (6.17), for all $\alpha, \beta \in \mathbb{N} \times \mathbb{N}$. Therefore, this polynomial is identically zero.

In particular, since $T_1(x) = x = 1 + (x-1)$ plays the role of the identity in the group $\{T_\alpha\}$, we have:

$$(T_\alpha)^{-1} = T_{1/\alpha}. \tag{6.39b}$$

Most of the classical formulae for Chebyshev polynomials survive the constructed extension – either by the argument given above to prove formula (6.39a), or by observing that

$$T_\alpha(x) = F\left(-\alpha, \alpha, \frac{1}{2}, \frac{1-x}{2}\right), \qquad \forall\, \alpha. \tag{6.40}$$

Remark 6.41. The theme of parameter extension is quite common in mathematics. The *methods* for constructing such extensions vary wildly. I chose in this Section the group

property of T_n's. A more direct approach is to extend the parameters' meaning in the hypergeometric representations, as in formulae (6.23) and (6.40). Then there are various *relations* one can extend. For example, the Legendre polynomials $P_n(x)$ enter into the following relations with the $T_n(x)$'s ([14, p. 261]):

$$\left(n + \frac{1}{2}\right)(1+x)^{1/2} \int_{-1}^{x} P_n(t)(x-t)^{-1/2}dt = T_n(x) + T_{n-1}(x), \tag{6.42a}$$

$$\left(n + \frac{1}{2}\right)(1-x)^{1/2} \int_{x}^{1} P_n(t)(t-x)^{-1/2}dt = T_n(x) - T_{n+1}(x), \tag{6.42b}$$

where $|x| < 1$, $n \in \mathbb{N}$, and the integrals are taken as Cauchy principal values.

Remark 6.43. Treatises and collections of formulae on special functions should always be read with a pen, pencil, and salt shaker ready. For example, the book [9] lists on p. 206 the following formula:

$$U_{n+1}(x) = (1-x)^{1/2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (n-k)!}{k!(n-2k)!}(2x)^{n-2k}. \tag{6.44}$$

Misprints are widespread, common, and not unexpected. Some errors are more subtle, and propagate from one book to another for *many* years. (See an example of 186 years old errors in [20]).

Remark 6.45. The Chebyshev polynomials $\{T_n(x)\}$ and the system $\{x^n\}$ are essentially the only two *polynomial* systems satisfying the semigroup/group law (see [18, 4]). More general systems, if any exist, could be found only in *series*. For example, if one restricts oneself to the hypergeometric (classical or basic) groups, one looks for triples of functions

$$a(x) = {}_p\mathcal{F}_q(\ldots; \omega_1(x-1)), \tag{6.46a}$$
$$b(x) = {}_r\mathcal{F}_s(\ldots; \omega_2(x-1)), \tag{6.46b}$$
$$c(x) = {}_P\mathcal{F}_Q(\ldots; \Omega(x-1)), \tag{6.46c}$$

such that

$$a(b(x)) = c(x). \tag{6.46d}$$

Remark 6.47. Suppose one quantizes the Chebyshev polynomials $U_n(x)$ by any of the standard methods of q-hypergeometric (= basic) series (see [2, 5, 6, 8].) Will then the sequences S_n and $S_n(q)$ constructed in § 2 appear as "values at $x = 1/2$" of the quantized polynomials? I don't see how, and think it is very unlikely, as one glance at formulae (5.25) and (5.27) will show. There is a deep mystery hidden here.

Remark 6.48. The proper meaning of the Chebyshev polynomials, as the generators of a semigroup of polynomical maps of a vector space, has been discovered by Veselov [22, 23] and by Hoffman and Withers [26, 10], with such semigroups parametrized by affine Weyl groups. It would be very interesting to find the analogs of the one-dimensional point $x = 1/2$ and periodic sequences, both classical and quantum, for this more general set-up.

I hope that posterity will judge me kindly,
not only as to the things which I have explained,
but also to those which I have intentionally
omitted so as to leave to others the pleasure
of discovery.

                Descartes

Acknowledgements

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